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The spherical-cap crack revisited

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Abstract

A crack in the shape of a spherical cap is subjected to a static loading. The exact solution for the crack-opening displacement is obtained using a method based on dual series equations and Laplace transforms. For shallow caps, the solution agrees with an asymptotic theory for perturbed penny-shaped cracks.

1. Introduction

Let Ω be a loaded crack in a three-dimensional elastic solid. Assume that Ω is bounded and that the crack edge $\partial\Omega$ is a simple, smooth, closed curve. The *form* of the stresses near $\partial\Omega$ is known (Leblond and Torlai, 1992), but the stress-intensity factors themselves can only be found by solving a boundary-value problem: they cannot be obtained from a local analysis near the edge.

There is only one non-planar Ω for which the boundary-value problem can be solved exactly, and that is a spherical cap. In fact, there are several Russian papers on this problem. One of the earliest is that of Ziuzin and Mossakovskii (1970), but their analysis was subsequently criticised by Prokhorova and Solov'ev (1976); both papers consider axisymmetric loadings, and use representations in terms of analytic functions of a complex variable. A method for non-axisymmetric problems was developed more recently by Popov (1992). He reduced the problems to some one-dimensional integral equations, whose desired solutions were shown to have non-integrable end-point singularities. A method based on dual series equations was sketched by Martynenko and Ulitko (1979). Their method is simple, in principle (we use it below), but it is, nevertheless, complicated when detailed results are required. A striking feature of these papers is that they do not contain mutual comparisons. Thus, it is difficult to know what the correct solution actually is, for any particular loading!

Consequently, we decided to re-work the problem. The main purpose of the calculation is to obtain a benchmark solution, so that other techniques (such as those based on solving a boundary integral equation numerically) can be validated. We have also confirmed the results for a *shallow* spherical cap by comparing with an asymptotic theory for perturbed penny-shaped cracks (Martin, 2000).

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The method used here is first described in the context of a model problem for Laplace's equation. It consists of patching separated solutions together in spherical polar coordinates, leading to some dual series equations, which are then reduced to an integro-differential equation. We solve this equation using Laplace transforms. We then generalise the method to the elasticity problem, which is much more complicated. It leads to a pair of coupled integro-differential equations, which are essentially those obtained previously by Martynenko and Ulitko (1979). We solve these using Laplace transforms. We also derive asymptotic approximations for the stress-intensity factors, valid for shallow spherical-cap cracks.

2. A model problem: potential flow past a rigid spherical cap

Before embarking on the elasticity problem, it is instructive to consider a simpler problem for Laplace's equation. The techniques used to solve this problem will generalise to the much more complicated crack problem. Thus, consider a rigid spherical cap, given by

$$r = c, \quad 0 \leqslant \theta < \alpha, \quad 0 \leqslant \phi < 2\pi,$$

where r, θ and ϕ are spherical polar coordinates. We want to solve Laplace's equation, $\nabla^2 u = 0$, outside the cap, with

$$\partial u/\partial r = -U \cos\theta \text{ on both sides of the cap,} \tag{2.1}$$

so that $Uz + u$ is the velocity potential for axisymmetric uniform flow past the cap; u is required to vanish as $r \to \infty$. Separation of variables gives the representations

$$u(r,\theta) = Uc \sum_{n=0}^{\infty} A_n (r/c)^n P_n(\cos\theta) \quad \text{for } 0 \leqslant r < c \quad \text{and}$$

$$u(r,\theta) = Uc \sum_{n=0}^{\infty} C_n (r/c)^{-n-1} P_n(\cos\theta) \quad \text{for } r > c,$$

where P_n is a Legendre polynomial and the dimensionless coefficients A_n and C_n are to be found. Continuity of $\partial u/\partial r$ across $r = c$ for $0 \leqslant \theta \leqslant \pi$ gives $nA_n = -(n+1)C_n$.

Define the discontinuity in u across $r = c$ by

$$[u(\theta)] = \lim_{r \to c-} u(r,\theta) - \lim_{r \to c+} u(r,\theta) = Uc \sum_{n=0}^{\infty} (2n+1)\mathscr{U}_n P_n(\cos\theta),$$

where $\mathscr{U}_n = A_n(n+1)^{-1}$. Using Eq. (2.1) and the fact that $[u(\theta)] = 0$ for $\alpha \leqslant \theta \leqslant \pi$ gives

$$\sum_{n=0}^{\infty} n(n+1)\mathscr{U}_n P_n(\cos\theta) = -\cos\theta, \quad 0 \leqslant \theta < \alpha, \tag{2.2}$$

$$\sum_{n=0}^{\infty} (2n+1)\mathscr{U}_n P_n(\cos\theta) = 0, \quad \alpha \leqslant \theta \leqslant \pi. \tag{2.3}$$

These form a pair of *dual series equations* for $\mathscr{U}_n$ (Sneddon, 1966). To solve them, we use an integral representation for $\mathscr{U}_n$,

$$\mathscr{U}_n = \frac{1}{2n+1} \int_0^\alpha \varphi(t) \sin\left(n + \frac{1}{2}\right) t \, \mathrm{d}t, \tag{2.4}$$

where $\varphi(t)$ is to be found. This representation ensures that Eq. (2.3) is satisfied for any φ, due to the discontinuous sum (A.4). It also gives

$$[u(\theta)] = Uc\int_\theta^\alpha \frac{\varphi(t)}{\sqrt{2\cos\theta - 2\cos t}}\,\mathrm{d}t, \quad 0 \leqslant \theta < \alpha.$$

From this, we can readily show that we must have $\varphi(0) = 0$, otherwise $[u(0)]$ would be unbounded.

Substitution of Eq. (2.4) in Eq. (2.2) gives

$$\sum_{n=0}^{\infty} \frac{n(n+1)}{2n+1}\int_0^\alpha \varphi(t)\,\sin\left(n+\frac{1}{2}\right)t\,\mathrm{d}t\,P_n(\cos\theta) = -\cos\theta, \quad 0 \leqslant \theta < \alpha,$$

an equation for $\varphi(t)$. To solve it, we would like to interchange the order of integration and summation, so as to obtain an integral equation for φ; however, the resulting sum is divergent, so that we must proceed indirectly. An integration by parts in Eq. (2.4) gives

$$2\lambda_n^2\mathscr{U}_n = \int_0^\alpha \varphi'(t)\,\cos\lambda_n t\,\mathrm{d}t - \varphi(\alpha)\,\cos\lambda_n\alpha, \tag{2.5}$$

where $\lambda_n = n + (1/2)$ and we have used $\varphi(0) = 0$. Then, we write Eq. (2.2) as

$$\sum_{n=0}^{\infty} 2\lambda_n^2\mathscr{U}_n P_n(\cos\theta) - \frac{1}{2}\sum_{n=0}^{\infty}\mathscr{U}_n P_n(\cos\theta) = -2\cos\theta, \quad 0 \leqslant \theta < \alpha.$$

Using Eqs. (2.4), (2.5), (A.3) and (A.5) gives

$$T_{t\to\theta}\left\{\varphi'(t) - \frac{1}{4}\int_t^\alpha \varphi(\tau)\,\mathrm{d}\tau\right\} = -2\cos\theta, \quad 0 \leqslant \theta < \alpha, \tag{2.6}$$

where we have defined the Abel operator T by

$$T\phi \equiv T_{t\to\theta}\{\phi(t)\} = \int_0^\theta \frac{\phi(t)\,\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}}. \tag{2.7}$$

This operator can be inverted: if $T\phi = f$, we can solve for ϕ as (Porter and Stirling, 1990, Section 9.2)

$$\phi(s) = T^{-1}f \equiv T^{-1}_{\theta\to s}\{f(\theta)\} = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \frac{f(\theta)\,\sin\theta}{\sqrt{2\cos\theta - 2\cos s}}\,\mathrm{d}\theta. \tag{2.8}$$

Applying T^{-1} to Eq. (2.6) gives

$$\varphi'(t) - \frac{1}{4}\int_t^\alpha \varphi(\tau)\,\mathrm{d}\tau = -(4/\pi)\cos\left(\frac{3}{2}t\right), \tag{2.9}$$

which is to be solved subject to $\varphi(0) = 0$.

One way to solve Eq. (2.9) is to differentiate with respect to t, leading to a second-order ordinary differential equation for φ with constant coefficients. The general solution of this equation contains two arbitrary constants; these are to be determined by imposing $\varphi(0) = 0$ and by requiring that the solution of the differential equation actually solves Eq. (2.9).

We shall use an alternative method, based on Laplace transforms. First, we write Eq. (2.9) in convolution form as

$$\varphi'(t) + \frac{1}{4}\int_0^t \varphi(\tau)\,\mathrm{d}\tau = \frac{1}{4}M_1 - (4/\pi)\cos\left(\frac{3}{2}t\right), \tag{2.10}$$

where $M_1 = \int_0^\alpha \varphi(t)\,\mathrm{d}t$ is an unknown constant. Next, we define

$$\Phi(p) = \mathscr{L}\{\varphi\} = \int_0^\infty \varphi(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t, \tag{2.11}$$

where p is the transform variable. Then, taking the Laplace transform of Eq. (2.10), we obtain

$$(p^2 + \tfrac{1}{4})\,\Phi(p) = \tfrac{1}{4}M_1 - (4/\pi)p^2/(p^2 + \tfrac{9}{4}),$$

whence

$$\Phi(p) = \left(\frac{M_1}{4} + \frac{1}{2\pi}\right)\frac{1}{p^2 + \frac{1}{4}} - \frac{9}{2\pi}\frac{1}{p^2 + \frac{9}{4}}.$$

Inverting, using $\mathscr{L}\{\sin\beta t\} = \beta/(p^2 + \beta^2)$, we obtain

$$\varphi(t) = \left(\tfrac{1}{2}M_1 + \pi^{-1}\right)\sin(\tfrac{1}{2}t) - (3/\pi)\sin(\tfrac{3}{2}t).$$

To determine M_1, we substitute for φ in the definition of M_1 and evaluate the integral, giving

$$M_1 = -(4/\pi)\,\sin\alpha\,\sin(\tfrac{1}{2}\alpha)$$

and then

$$\varphi(t) = (1/\pi)\,\sec(\tfrac{1}{2}\alpha)\,\cos(\tfrac{3}{2}\alpha)\,\sin(\tfrac{1}{2}t) - (3/\pi)\,\sin(\tfrac{3}{2}t).$$

This solution agrees with the well known solution of Collins (1959).

3. Elastic field representations

For the crack problem, we start with representations for the elastic displacement $\boldsymbol{u}$ in terms of potentials. From Lur'e (1964, Section 6.2), we have general solutions for three-dimensional elasticity, in spherical polar coordinates (r, θ, ϕ), where $\boldsymbol{u} = (u_r, u_\theta, u_\phi)$ is independent of the azimuthal angle, ϕ. Thus, the following representations can be obtained.

Interior solution: displacements

$$u_r = \{Ar^{n+1}(n+1)(n-2+4v) + Bnr^{n-1}\}P_n,$$
$$u_\theta = -\{Ar^{n+1}(n+5-4v) + Br^{n-1}\}P_n^1.$$

Exterior solution: displacements

$$u_r = \{Cr^{-n}n(n+3-4v) - Dr^{-n-2}(n+1)\}P_n,$$
$$u_\theta = \{Cr^{-n}(n-4+4v) - Dr^{-n-2}\}P_n^1.$$

Interior solution: stresses

$$(2\mu)^{-1}\tau_{rr} = \{A(n+1)(n^2-n-2-2v)r^n + Bn(n-1)r^{n-2}\}P_n,$$
$$(2\mu)^{-1}\tau_{r\theta} = -\{A(n^2+2n-1+2v)r^n + B(n-1)r^{n-2}\}P_n^1.$$

Exterior solution: stresses

$$(2\mu)^{-1}\tau_{rr} = -\{Cn(n^2+3n-2v)r^{-n-1} - D(n+1)(n+2)r^{-n-3}\}P_n,$$
$$(2\mu)^{-1}\tau_{r\theta} = -\{C(n^2-2+2v)r^{-n-1} - D(n+2)r^{-n-3}\}P_n^1.$$

All these solutions are valid for $n = 0, 1, 2, \ldots$. In them, v is Poisson's ratio (Lur'e uses $m = 1/v$), μ is the shear modulus, $P_n = P_n(\cos\theta)$ and $P_n^1 = P_n^1(\cos\theta) = P_n'(\cos\theta)\sin\theta = -(\mathrm{d}/\mathrm{d}\theta)P_n(\cos\theta)$; note that $P_0^1 \equiv 0$. Expressions for the other stress components are given on p. 330 of Lur'e's book.

We consider a crack in the shape of a spherical cap, given by $r = c$, $0 \leqslant \theta < \alpha$ and $0 \leqslant \phi < 2\pi$. Using superscripts (1) and (2) for the regions $r < c$ and $r > c$, respectively, we have the following representations for the displacements and stresses:

$$u_r^{(1)} = A_0(4v-2)r + c\sum_{n=1}^{\infty}\left[A_n\left(\frac{r}{c}\right)^{n+1}(n+1)(n-2+4v) + B_n\left(\frac{r}{c}\right)^{n-1}n\right]P_n, \tag{3.1}$$

$$u_r^{(2)} = -cD_0\left(\frac{c}{r}\right)^2 + c\sum_{n=1}^{\infty}\left[C_n\left(\frac{c}{r}\right)^n n(n+3-4v) - D_n\left(\frac{c}{r}\right)^{n+2}(n+1)\right]P_n, \tag{3.2}$$

$$u_\theta^{(1)} = -c\sum_{n=1}^{\infty}\left[A_n\left(\frac{r}{c}\right)^{n+1}(n+5-4v) + B_n\left(\frac{r}{c}\right)^{n-1}\right]P_n^1, \tag{3.3}$$

$$u_\theta^{(2)} = c\sum_{n=1}^{\infty}\left[C_n\left(\frac{c}{r}\right)^n(n-4+4v) - D_n\left(\frac{c}{r}\right)^{n+2}\right]P_n^1, \tag{3.4}$$

$$\begin{aligned}(2\mu)^{-1}\tau_{rr}^{(1)} = &-2(1+v)A_0 \\ &+\sum_{n=1}^{\infty}\left[A_n\left(\frac{r}{c}\right)^n(n+1)(n^2-n-2-2v) + B_n\left(\frac{r}{c}\right)^{n-2}n(n-1)\right]P_n,\end{aligned} \tag{3.5}$$

$$(2\mu)^{-1}\tau_{rr}^{(2)} = 2D_0\left(\frac{c}{r}\right)^3 - \sum_{n=1}^{\infty}\left[C_n\left(\frac{c}{r}\right)^{n+1}n(n^2+3n-2v) - D_n\left(\frac{c}{r}\right)^{n+3}(n+1)(n+2)\right]P_n,$$

$$(2\mu)^{-1}\tau_{r\theta}^{(1)} = -\sum_{n=1}^{\infty}\left[A_n\left(\frac{r}{c}\right)^n(n^2+2n-1+2v) + B_n\left(\frac{r}{c}\right)^{n-2}(n-1)\right]P_n^1, \tag{3.6}$$

$$(2\mu)^{-1}\tau_{r\theta}^{(2)} = -\sum_{n=1}^{\infty}\left[C_n\left(\frac{c}{r}\right)^{n+1}(n^2-2+2v) - D_n\left(\frac{c}{r}\right)^{n+3}(n+2)\right]P_n^1.$$

These formulas were obtained using $A = c^{-n}A_n$, $B = c^{2-n}B_n$, $C = c^{n+1}C_n$ and $D = c^{n+3}D_n$, where the coefficients A_n, B_n, C_n and D_n are dimensionless. These coefficients are to be determined by applying the boundary conditions on $r = c$. The first of these is that the stresses should be continuous across $r = c$ for all θ, when

$$-(1+v)A_0 = D_0, \tag{3.7}$$

$$A_n(n+1)(n^2-n-2-2v) + B_n n(n-1) = -C_n n(n^2+3n-2v) + D_n(n+1)(n+2), \tag{3.8}$$

$$A_n(n^2+2n-1+2v) + B_n(n-1) = C_n(n^2-2+2v) - D_n(n+2), \tag{3.9}$$

for $n = 1, 2, \ldots$. From these, we obtain

$$\Delta_n C_n = (n-1)\{(n+1)(2n+3)A_n + (2n+1)B_n\}, \tag{3.10}$$

$$(n+2)\Delta_n D_n = (2n+1)\{n(n+2)(n^2-1) + 4 - 4v^2\}A_n + n(n-1)(n+2)(2n-1)B_n, \tag{3.11}$$

where $\Delta_n = -2\{n^2 - n + (2n+1)(1-v)\}$. In particular, $C_1 = 0$. Note that Eq. (3.11) reduces to Eq. (3.7) when $n = 0$.

If the material in $r < c$ is different from that in $r > c$, one obtains an *interface crack* on $r = c$. The corresponding problem has been considered by Altenbach et al. (1995), using the representations given above.

4. The crack-opening displacement

We define the crack-opening displacement $[\boldsymbol{u}]$ by

$$[u_r(\theta)] = u_r^{(1)}(c,\theta) - u_r^{(2)}(c,\theta) = c\sum_{n=0}^{\infty}(2n+1)\mathscr{U}_n P_n, \tag{4.1}$$

$$[u_\theta(\theta)] = u_\theta^{(1)}(c,\theta) - u_\theta^{(2)}(c,\theta) = c\sum_{n=1}^{\infty}(2n+1)\mathscr{V}_n P_n^1, \tag{4.2}$$

where the dimensionless coefficients $\mathscr{U}_n$ and $\mathscr{V}_n$ are to be determined. Comparing with the representations (3.1)–(3.4), we find that $\mathscr{U}_0 = (4\nu - 2)A_0 + D_0 = -3(1-\nu)A_0$,

$$\begin{aligned}(2n+1)\mathscr{U}_n &= A_n(n+1)(n-2+4\nu) + B_n n - C_n n(n+3-4\nu) + D_n(n+1),\\ -(2n+1)\mathscr{V}_n &= A_n(n+5-4\nu) + B_n + C_n(n-4+4\nu) - D_n.\end{aligned}$$

Substituting for C_n and D_n from Eqs. (3.10) and (3.11), respectively, we find that

$$\begin{aligned}(n+2)\Delta_n\mathscr{U}_n &= -2(1-\nu)\{(2n+3)(n^2-2+2\nu)(n+1)A_n + (2n-1)(n+2)nB_n\},\\ (n+2)\Delta_n\mathscr{V}_n &= 2(1-\nu)\{(2n+3)(n^2+3n-2\nu)A_n + (2n-1)(n+2)B_n\}.\end{aligned}$$

We can rewrite these, giving A_n and B_n in terms of $\mathscr{U}_n$ and $\mathscr{V}_n$:

$$\begin{aligned}2(1-\nu)(2n+3)A_n &= -(n+2)(\mathscr{U}_n + n\mathscr{V}_n),\\ 2(1-\nu)(2n-1)B_n &= (n^2+3n-2\nu)\mathscr{U}_n + (n+1)(n^2-2+2\nu)\mathscr{V}_n.\end{aligned}$$

We can now state the problem to be solved: find $\mathscr{U}_n$ and $\mathscr{V}_n$, so that

$$[u_r(\theta)] = 0 \quad \text{and} \quad [u_\theta(\theta)] = 0 \quad \text{for } \alpha < \theta \leqslant \pi, \tag{4.3}$$

and

$$\tau_{rr}(\theta) = -\mu q_r(\theta) \quad \text{and} \quad \tau_{r\theta}(\theta) = -\mu q_\theta(\theta) \quad \text{on } r = c \quad \text{for } 0 \leqslant \theta < \alpha, \tag{4.4}$$

where q_r and q_θ are given functions of θ. In particular, for uniaxial tension at infinity in the z-direction (so that $\tau_{zz}^{\infty} = p_0$, say), we have (Lur'e, 1964, p. 343)

$$q_r(\theta) = (p_0/\mu)\cos^2\theta \quad \text{and} \quad q_\theta(\theta) = -(p_0/\mu)\sin\theta\cos\theta. \tag{4.5}$$

Using Eq. (4.4) in Eqs. (3.5) and (3.6) gives

$$\begin{aligned}-\frac{1}{2}q_r &= -2(1+\nu)A_0 + \sum_{n=1}^{\infty}\{A_n(n+1)(n^2-n-2-2\nu) + B_n n(n-1)\}P_n,\\ \frac{1}{2}q_\theta &= \sum_{n=1}^{\infty}\{A_n(n^2+2n-1+2\nu) + B_n(n-1)\}P_n^1.\end{aligned}$$

Eliminating A_n and B_n in favour of $\mathscr{U}_n$ and $\mathscr{V}_n$ gives

$$-\frac{1}{2}(1-\nu)q_r = \sum_{n=0}^{\infty}\{2[w_n^2 - 1 + \nu(2w_n - 1)]\mathscr{U}_n + w_n[w_n + 1 + \nu(4w_n - 5)]\mathscr{V}_n\}\frac{P_n}{4w_n - 3}, \tag{4.6}$$

$$-\frac{1}{2}(1-\nu)q_\theta = \sum_{n=1}^{\infty}\{[w_n + 1 + \nu(4w_n - 5)]\mathscr{U}_n + [(w_n + 1)(2w_n - 3) + 3\nu]\mathscr{V}_n\}\frac{P_n^1}{4w_n - 3} \tag{4.7}$$

for $0 \leqslant \theta < \alpha$, where $w_n = n(n+1)$. These are to be solved subject to Eq. (4.3), wherein $[\boldsymbol{u}]$ is defined by Eqs. (4.1) and (4.2). Of particular interest are the stress-intensity factors. We define these by

$$[u_r] \sim K_n\sqrt{2a}\sqrt{c(\alpha-\theta)} \quad \text{and} \quad [u_\theta] \sim K_s\sqrt{2a}\sqrt{c(\alpha-\theta)} \quad \text{as } \theta \to \alpha-, \tag{4.8}$$

where a is a length scale and K_n and K_s are dimensionless stress-intensity factors. It is convenient to take $a = c\sin\alpha$, for we may then consider the limiting case of a penny-shaped crack of radius a, obtained by taking the limits $c \to \infty$ and $\alpha \to 0$ with a fixed.

The definitions of the stress-intensity factors in Eq. (4.8) are convenient, but not standard. For example, it is usual to suppose that $\tau_{rr} \sim \widetilde{K}_n/\sqrt{2\pi\rho'}$ as $\rho' \to 0$, where ρ' is distance from the crack edge $\partial\Omega$. Making use of the known general relations between $[\boldsymbol{u}]$ behind $\partial\Omega$ and the stress components ahead of $\partial\Omega$ (see, for example, Rice (1989, p. 32)), we find that

$$\widetilde{K}_n = \tfrac{1}{2}\mu\sqrt{\pi a}K_n/(1-\nu). \tag{4.9}$$

This formula and the corresponding formula for K_s can be used to obtain expressions for the standard stress-intensity factors from the results derived below.

5. Reduction to integro-differential equations

We introduce representations (2.4) for $\mathscr{U}_n$ and

$$\mathscr{V}_n = \frac{1}{4n(n+1)}\int_0^\alpha \psi(t)\cos\left(n+\frac{1}{2}\right)t\,\mathrm{d}t \tag{5.1}$$

for $\mathscr{V}_n$, where the functions φ and ψ are to be found. Substituting Eq. (2.4) in Eq. (4.1), followed by evaluation of the sum using Eq. (A.4), shows that $[u_r(\theta)] = 0$ for $\theta > \alpha$, as required, for any choice of φ; we also obtain

$$[u_r(\theta)] = c\int_\theta^\alpha \frac{\varphi(t)}{\sqrt{2\cos\theta - 2\cos t}}\,\mathrm{d}t, \quad 0 \leqslant \theta < \alpha. \tag{5.2}$$

$[u_r(0)]$ will be bounded provided that $t^{-1}\varphi(t)$ is integrable near $t = 0$, so that, in particular,

$$\varphi(0) = 0. \tag{5.3}$$

Similarly, substituting Eq. (5.1) in Eq. (4.2), followed by use of Eq. (A.7) shows that $[u_\theta(\theta)] = 0$ for $\theta > \alpha$, as required, provided that ψ satisfies

$$\int_0^\alpha \psi(t)\cos\left(\frac{1}{2}t\right)\mathrm{d}t = 0. \tag{5.4}$$

We also obtain

$$[u_\theta(\theta)] = \frac{-c}{2\sin\theta}\int_\theta^\alpha \frac{\psi(t)\sin t}{\sqrt{2\cos\theta - 2\cos t}}\,\mathrm{d}t, \quad 0 \leqslant \theta < \alpha; \tag{5.5}$$

it turns out that $[u_\theta(0)] = 0$, as expected from symmetry considerations.

The stress-intensity factors, K_n and K_s, can be expressed directly in terms of φ and ψ, respectively. Thus, from Eq. (5.2), we have

$$[u_r(\theta)] \sim c\,\varphi(\alpha)\,(\sin\alpha)^{-1}\sqrt{2\cos\theta - 2\cos\alpha} \quad \text{as } \theta \to \alpha-$$

with a similar approximation for $[u_\theta]$. Then, comparison with Eq. (4.8) gives

$$K_n = \frac{\varphi(\alpha)}{\sin\alpha} \quad \text{and} \quad K_s = \frac{-\psi(\alpha)}{2\sin\alpha}. \tag{5.6}$$

To obtain φ and ψ, we substitute Eqs. (2.4) and (5.1) in Eqs. (4.6) and (4.7), and evaluate the series using results from the appendix. To do this, we note the following partial-fraction expansions:

$$\frac{2[w_n^2 - 1 + v(2w_n - 1)]}{(2n+1)(4w_n - 3)} = \frac{\lambda_n}{4} + \frac{\delta_1}{4\lambda_n} - \frac{\delta_3}{8}\left(\frac{1}{\lambda_n - 1} + \frac{1}{\lambda_n + 1}\right), \tag{5.7}$$

$$\frac{w_n + 1 + v(4w_n - 5)}{4w_n - 3} = \delta_0 + \frac{\delta_3}{2}\left(\frac{1}{\lambda_n - 1} - \frac{1}{\lambda_n + 1}\right), \tag{5.8}$$

$$\frac{w_n + 1 + v(4w_n - 5)}{(2n+1)(4w_n - 3)} = \frac{\delta_0\lambda_n}{2w_n} + \frac{\delta_2}{8w_n\lambda_n} + \frac{3\delta_3}{16w_n}\left(\frac{1}{\lambda_n - 1} + \frac{1}{\lambda_n + 1}\right), \tag{5.9}$$

$$\frac{(w_n + 1)(2w_n - 3) + 3v}{4w_n - 3} = \frac{\lambda_n^2}{2} - \frac{3\delta_3}{4}\left(\frac{1}{\lambda_n - 1} - \frac{1}{\lambda_n + 1}\right). \tag{5.10}$$

In these, $w_n = n(n+1)$, $\lambda_n = n + 1/2$,

$$\delta_0 = \tfrac{1}{4}(1 + 4v), \quad \delta_1 = \tfrac{3}{16}(5 + 8v), \quad \delta_2 = \tfrac{3}{16}(1 - 8v) \quad \text{and} \quad \delta_3 = \tfrac{1}{16}(7 - 8v).$$

However, prior to substitution, we note that the first terms on the right-hand sides of Eqs. (5.7) and (5.10) will lead to divergent series. To overcome this, we first integrate by parts in Eqs. (2.4) and (5.1). Thus, making use of $\varphi(0) = 0$, we obtain Eq. (2.5) and then

$$\sum_{n=0}^{\infty}(2n+1)\mathscr{U}_n\,\frac{\lambda_n}{4}\,P_n(\cos\theta) = \frac{1}{4}\int_0^{\alpha}\varphi'(t)\,f_0(t;\theta)\,\mathrm{d}t - \frac{1}{4}\,\varphi(\alpha)\,f_0(\alpha;\theta)$$
$$= \frac{1}{4}\int_0^{\theta}\frac{\varphi'(t)\,\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}},$$

where f_0 is defined by Eq. (A.2) and we have used Eq. (A.3), noting that $\theta < \alpha$. Similarly, we have

$$4w_n\lambda_n\mathscr{V}_n = \psi(\alpha)\,\sin\lambda_n\alpha - \int_0^{\alpha}\psi'(t)\,\sin\lambda_n t\,\mathrm{d}t,$$

whence

$$\sum_{n=1}^{\infty}\mathscr{V}_n\,\frac{\lambda_n^2}{2}\,P_n^1(\cos\theta) = \frac{-1}{8\sin\theta}\int_0^{\theta}\frac{\psi'(t)\,\sin t\,\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}},$$

where we have used Eqs. (A.6) and (5.4). Use of these results, together with Eqs. (2.4), (5.1) and (5.7)–(5.10), in Eqs. (4.6) and (4.7) gives

$$-\frac{1}{2}(1 - v)q_r = \frac{1}{4}\,T_{t\to\theta}\{\varphi'(t)\} + \int_0^{\alpha}\{\varphi(t)\,S_{11}(t;\theta) + \psi(t)\,S_{12}(t;\theta)\}\mathrm{d}t, \tag{5.11}$$

$$-\frac{1}{2}(1 - v)q_\theta = \frac{-1}{8\sin\theta}\,T_{t\to\theta}\{\psi'(t)\,\sin t\} + \int_0^{\alpha}\{\varphi(t)\,S_{21}(t;\theta) + \psi(t)\,S_{22}(t;\theta)\}\mathrm{d}t, \tag{5.12}$$

for $0 \leqslant \theta < \alpha$, where

$$S_{11} = \sum_{n=0}^{\infty} \left\{ \frac{\delta_1}{4\lambda_n} - \frac{\delta_3}{8} \left(\frac{1}{\lambda_n - 1} + \frac{1}{\lambda_n + 1} \right) \right\} P_n(\cos\theta)\, \sin\lambda_n t,$$

$$S_{12} = \sum_{n=0}^{\infty} \left\{ \frac{\delta_0}{4} + \frac{\delta_3}{8} \left(\frac{1}{\lambda_n - 1} - \frac{1}{\lambda_n + 1} \right) \right\} P_n(\cos\theta)\, \cos\lambda_n t,$$

$$S_{21} = \sum_{n=1}^{\infty} \left\{ \frac{\delta_0 \lambda_n}{2} + \frac{\delta_2}{8\lambda_n} + \frac{3\delta_3}{16} \left(\frac{1}{\lambda_n - 1} + \frac{1}{\lambda_n + 1} \right) \right\} P_n^1\, \frac{\sin\lambda_n t}{w_n},$$

$$S_{22} = -\frac{3\delta_3}{16} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n - 1} - \frac{1}{\lambda_n + 1} \right) P_n^1(\cos\theta)\, \frac{\cos\lambda_n t}{w_n}.$$

The Abel operator T and its inverse are defined by Eqs. (2.7) and (2.8), respectively.

The sums S_{ij} can be evaluated explicitly using results from the appendix. We obtain

$$S_{11} = \frac{1}{4} \int_0^t f_0(\tau;\theta)\, \{\delta_1 - \delta_3 \cos(t-\tau)\}\, d\tau + \frac{\delta_3}{2} \sin t\, \sin\left(\frac{1}{2}\theta\right),$$

$$S_{12} = \frac{\delta_0}{4} f_0(t;\theta) - \frac{\delta_3}{4} \int_0^t f_0(\tau;\theta)\, \sin(t-\tau)\, d\tau - \frac{\delta_3}{2} \cos t\, \sin\left(\frac{1}{2}\theta\right),$$

$$\sin\theta\, S_{21} = \tfrac{1}{2}\delta_0 f_0(t;\theta)\, \sin t - \delta_3 \sin^3(\tfrac{1}{2}\theta)\, \sin t + \frac{1}{8} \int_0^t f_0^1(\tau;\theta)\, \{\delta_2 + 3\delta_3 \cos(t-\tau)\}\, d\tau,$$

$$\sin\theta\, S_{22} = \frac{3\delta_3}{8} \int_0^t f_0^1(\tau;\theta)\, \sin(t-\tau)\, d\tau - \delta_3 \sin^2\left(\frac{1}{2}\theta\right) \left(\cos\left(\frac{1}{2}t\right) - \sin\left(\frac{1}{2}\theta\right) \cos t \right),$$

where f_0^1 is defined by Eq. (A.8). Then, changing the order of integration gives

$$\int_0^\alpha \varphi(t)\, S_{11}(t;\theta)\, dt = \frac{1}{4}\, T_{t\to\theta} \left\{ \int_t^\alpha [\delta_1 - \delta_3 \cos(t-\tau)]\, \varphi(\tau)\, d\tau \right\} + \frac{\delta_3}{2} \sin\left(\frac{1}{2}\theta\right) \int_0^\alpha \varphi(t)\, \sin t\, dt,$$

$$\int_0^\alpha \psi(t)\, S_{12}(t;\theta)\, dt = \frac{1}{4}\, T_{t\to\theta} \left\{ \delta_0 \psi(t) + \delta_3 \int_t^\alpha \psi(\tau)\, \sin(t-\tau)\, d\tau \right\} - \frac{\delta_3}{2} \sin\left(\frac{1}{2}\theta\right) \int_0^\alpha \psi(t)\, \cos t\, dt,$$

$$\sin\theta \int_0^\alpha \varphi(t)\, S_{21}(t;\theta)\, dt = \frac{1}{8}\, \widehat{T}_{t\to\theta} \left\{ \int_t^\alpha [\delta_2 + 3\delta_3 \cos(t-\tau)]\, \varphi(\tau)\, d\tau \right\} + \frac{1}{2} \delta_0\, T_{t\to\theta}\{\varphi(t)\, \sin t\} - \delta_3 \sin^3\left(\frac{1}{2}\theta\right) \int_0^\alpha \varphi(t)\, \sin t\, dt,$$

$$\sin\theta \int_0^\alpha \psi(t)\, S_{22}(t;\theta)\, dt = -\frac{3\delta_3}{8}\, \widehat{T}_{t\to\theta} \left\{ \int_t^\alpha \psi(\tau)\, \sin(t-\tau)\, d\tau \right\} + \delta_3 \sin^3\left(\frac{1}{2}\theta\right) \int_0^\alpha \psi(t)\, \cos t\, dt,$$

where we have used Eq. (5.4) to obtain the last formula and the operator $\widehat{T}$ is defined by

$$\widehat{T}\phi \equiv \widehat{T}_{t\to\theta}\{\phi(t)\} = \int_0^\theta \phi(t)\, \sqrt{2\cos t - 2\cos\theta}\, dt.$$

Using these results, we apply $4T^{-1}$ to Eq. (5.11), giving

$$\varphi'(s) + \delta_0\psi(s) + \delta_3 \sin s \int_0^\alpha \{\varphi(t)\sin t - \psi(t)\cos t\}\,\mathrm{d}t + \int_s^\alpha \{\delta_1 - \delta_3\cos(s-t)\}\,\varphi(t)\,\mathrm{d}t + \delta_3 \int_s^\alpha \psi(t)\sin(s-t)\,\mathrm{d}t = Q_r(s) \tag{5.13}$$

for $0 < s < \alpha$, where $Q_r(s) = -2(1-\nu)\,T^{-1}_{\theta\to s}\{q_r(\theta)\}$ and we have used $T^{-1}_{\theta\to s}\{\sin\frac{1}{2}\theta\} = (1/2)\sin s$.

Next, we apply $-8(\sin s)^{-1}\,T^{-1}\sin\theta$ to Eq. (5.12). To do this, we first calculate that

$$T^{-1}_{\theta\to s}\widehat{T}_{t\to\theta}\{\phi(t)\} = \sin s \int_0^s \phi(t)\,\mathrm{d}t,$$

and $T^{-1}_{\theta\to s}\{\sin^3\frac{1}{2}\theta\} = \frac{3}{4}\sin s\,\sin^2(\frac{1}{2}s)$, whence

$$\psi'(s) - 4\delta_0\varphi(s) - \delta_2\left\{\int_0^s \varphi(t)\,t\,\mathrm{d}t + s\int_s^\alpha \varphi(t)\,\mathrm{d}t\right\} - 3\delta_3\cos s\int_0^\alpha \{\varphi(t)\sin t - \psi(t)\cos t\}\,\mathrm{d}t - 3\delta_3\int_0^s \psi(t)\,\mathrm{d}t - 3\delta_3\int_s^\alpha \{\varphi(t)\sin(s-t) + \psi(t)\cos(s-t)\}\,\mathrm{d}t = Q_\theta(s), \tag{5.14}$$

for $0 < s < \alpha$, where $Q_\theta(s) = 4(1-\nu)(\sin s)^{-1}\,T^{-1}_{\theta\to s}\{q_\theta(\theta)\sin\theta\}$.

Eqs. (5.13) and (5.14) have been obtained by Martynenko and Ulitko (1979), although few details were given; see their Eq. (2.4), noting that our ψ is their (-4ψ). These authors considered all-round tension at infinity.

For uniaxial tension at infinity, q_r and q_θ are given by Eq. (4.5); elementary calculations, using Eq. (2.8), then give

$$Q_r(s) = \tfrac{1}{3}\eta(\cos(\tfrac{1}{2}s) + 2\cos(\tfrac{5}{2}s)) \quad \text{and} \quad Q_\theta(s) = \tfrac{8}{5}\eta\sin(\tfrac{5}{2}s), \tag{5.15}$$

with $\eta = -4(1-\nu)p_0/(\pi\mu)$.

6. Solution of the integro-differential equations

Martynenko and Ulitko (1979) solved Eqs. (5.13) and (5.14) (for all-round tension) by repeated differentiation, leading to a pair of coupled homogeneous ordinary differential equations with constant coefficients, relating φ, φ'', $\varphi^{(\mathrm{iv})}$, ψ', ψ''' and $\psi^{(\mathrm{v})}$. The general solution of this system contains 17 arbitrary constants, and these were then determined by substituting back in Eqs. (5.13) and (5.14), together with the conditions (5.3) and (5.4).

Here, we solve Eqs. (5.13) and (5.14) using Laplace transforms. First, we rewrite the integro-differential equations in convolution form as

$$\varphi'(s) + \delta_0\psi(s) - \int_0^s \{k_{11}(s-t)\,\varphi(t) + k_{12}(s-t)\,\psi(t)\}\mathrm{d}t = g_1(s), \tag{6.1}$$

$$\psi'(s) - 4\delta_0\varphi(s) - \int_0^s \{k_{21}(s-t)\,\varphi(t) + k_{22}(s-t)\,\psi(t)\}\mathrm{d}t = g_2(s), \tag{6.2}$$

where

$$k_{11}(\tau) = \delta_1 - \delta_3\cos\tau, \quad k_{12}(\tau) = \delta_3\sin\tau,$$
$$k_{21}(\tau) = -\delta_2\tau - 3\delta_3\sin\tau, \quad k_{22}(\tau) = 3\delta_3(1-\cos\tau),$$
$$g_1(s) = Q_r(s) + \delta_3 M_2\cos s - \delta_1 M_1, \quad g_2(s) = Q_\theta(s) + 3\delta_3 M_2\sin s + s\delta_2 M_1,$$

$$M_1 = \int_0^\alpha \varphi(t)\,\mathrm{d}t \quad \text{and} \quad M_2 = \int_0^\alpha \{\varphi(t)\cos t + \psi(t)\sin t\}\,\mathrm{d}t. \tag{6.3}$$

Note that the constants M_1 and M_2 are unknown, as they depend on the solutions φ and ψ.

Next, we introduce $\Phi(p) = \mathscr{L}\{\varphi\}$ and $\Psi(p) = \mathscr{L}\{\psi\}$, the Laplace transforms of φ and ψ, respectively, defined by Eq. (2.11); we use upper-case letters to denote the Laplace transforms of other functions. Then, taking the Laplace transform of Eqs. (6.1) and (6.2) gives

$$(p - K_{11})\Phi + (\delta_0 - K_{12})\Psi = G_1,$$
$$-(4\delta_0 + K_{21})\Phi + (p - K_{22})\Psi = G_2 + \psi_0,$$

where $\psi_0 = \psi(0)$ and we have used $\varphi(0) = 0$. Hence

$$\Delta\Phi = (p - K_{22})G_1 - (\delta_0 - K_{12})(G_2 + \psi_0), \tag{6.4}$$
$$\Delta\Psi = (4\delta_0 + K_{21})G_1 + (p - K_{11})(G_2 + \psi_0), \tag{6.5}$$

where $\Delta = (p - K_{11})(p - K_{22}) + (\delta_0 - K_{12})(4\delta_0 + K_{21})$. We have

$$K_{11}(p) = \frac{\delta_1}{p} - \frac{\delta_3 p}{p^2+1}, \quad K_{12}(p) = \frac{\delta_3}{p^2+1},$$
$$K_{21}(p) = -\frac{\delta_2}{p^2} - \frac{3\delta_3}{p^2+1}, \quad K_{22}(p) = \frac{3\delta_3}{p(p^2+1)},$$

whence

$$\Delta(p) = \frac{(p^2 + \frac{9}{4})R(p^2)}{p^2(p^2+1)}, \tag{6.6}$$

where

$$R(x) = x^2 - \tfrac{1}{2}(3 - 8v^2)x + \tfrac{9}{16}.$$

Note that $R(x) = 0$ when $x = \frac{3}{4} - 2v^2 \pm 2\mathrm{i}v\gamma$ with $\gamma = \frac{1}{2}\sqrt{3 - 4v^2}$.

From Eqs. (6.4)–(6.6), we obtain

$$\Phi(p) = \frac{[p^2(p^2+1) - 3\delta_3]pG_1 - [\delta_0 p^2 - \delta_2]p^2(G_2 + \psi_0)}{(p^2 + \frac{9}{4})R(p^2)},$$
$$\Psi(p) = \frac{4(\delta_0 p^2 - \delta_2)(p^2 + \frac{1}{4})G_1 + [p^4 + (1 - 2\delta_0)p^2 - \delta_1]p(G_2 + \psi_0)}{(p^2 + \frac{9}{4})R(p^2)},$$

using $\delta_3 = \delta_0 + \delta_2 = \delta_1 - 2\delta_0$. Also, for uniaxial loading, Eq. (5.15) gives

$$G_1(p) = \eta\frac{p(p^2 + \frac{9}{4})}{(p^2 + \frac{1}{4})(p^2 + \frac{25}{4})} + \frac{\delta_3 M_2 p^2 - \delta_1 M_1(p^2+1)}{p(p^2+1)},$$
$$G_2(p) = \eta\frac{4}{p^2 + \frac{25}{4}} + \frac{3\delta_3 M_2 p^2 + \delta_2 M_1(p^2+1)}{p^2(p^2+1)}.$$

Thus, it is convenient to write

$$\Phi = \eta\Phi_1 + \Phi_2 \quad \text{and} \quad \Psi = \eta\Psi_1 + \Psi_2.$$

We have

$$\Phi_1 = \frac{p^2\mathscr{A}_1(p^2)}{(p^2 + \frac{1}{4})(p^2 + \frac{9}{4})(p^2 + \frac{25}{4})R(p^2)} \quad \text{and} \quad \Phi_2 = \frac{\mathscr{A}_2(p^2)}{(p^2 + \frac{9}{4})R(p^2)},$$

where

$$\mathscr{A}_1(p^2) = p^6 + \tfrac{1}{4}p^4(9 - 16\nu) + \tfrac{1}{16}p^2(23 - 88\nu) - \tfrac{1}{64}(177 - 120\nu),$$
$$\mathscr{A}_2(p^2) = p^2(p^2 - 3\delta_0)\delta_3 M_2 - p^2(\delta_0 p^2 - \delta_2)\psi_0 - \{p^4\delta_1 + p^2(\delta_1 + \delta_0\delta_2) - \tfrac{81}{64}\}M_1.$$

Next, we split into partial fractions, as

$$\Phi_1 = \frac{A_1}{p^2 + \frac{1}{4}} + \frac{B_1}{p^2 + \frac{9}{4}} + \frac{C_1}{p^2 + \frac{25}{4}} + \frac{(p^2 + \frac{3}{4})D_1 + (p^2 - \frac{3}{4})E_1}{R(p^2)},$$
$$\Phi_2 = \frac{B_2}{p^2 + \frac{9}{4}} + \frac{(p^2 + \frac{3}{4})D_2 + (p^2 - \frac{3}{4})E_2}{R(p^2)},$$

where

$$A_1 = \frac{1}{16(1 + \nu)}, \quad B_1 = \frac{-3}{16(1 - \nu)}, \quad C_1 = \frac{25}{4(7 - 5\nu)},$$

$$D_1 = \frac{-\nu(3 + 5\nu - 10\nu^2)}{4(1 - \nu^2)(7 - 5\nu)}, \qquad E_1 = \frac{13 - 25\nu - 6\nu^2 + 20\nu^3}{8(1 - \nu^2)(7 - 5\nu)},$$

$$B_2 = \frac{3}{64(1 - \nu)}\{16\delta_3 M_2 - 4\psi_0 - (3 + 8\nu)M_1\},$$
$$D_2 = \frac{-\nu}{32(1 - \nu)}\{16\delta_3 M_2 + 4(3 - 4\nu)\psi_0 + (13 - 24\nu)M_1\},$$
$$E_2 = \frac{1}{64(1 - \nu)}\left\{16\delta_3(1 - 2\nu)M_2 - 4(1 + 6\nu - 8\nu^2)\psi_0 - (51 - 14\nu - 48\nu^2)M_1\right\}.$$

Note that

$$\lim_{p\to\infty} \Phi_1(p) = 1 = A_1 + B_1 + C_1 + D_1 + E_1. \tag{6.7}$$

Inverting the Laplace transforms, we obtain

$$\varphi(t) = 2\eta A_1 \sin(\tfrac{1}{2}t) + \tfrac{2}{3}(\eta B_1 + B_2)\sin(\tfrac{3}{2}t) + \tfrac{2}{5}\eta C_1 \sin(\tfrac{5}{2}t) + \nu^{-1}(\eta D_1 + D_2)\cosh\gamma t \sin \nu t$$
$$+ \gamma^{-1}(\eta E_1 + E_2)\sinh\gamma t \cos \nu t, \tag{6.8}$$

using $\mathscr{L}\{\cosh\gamma t \sin \nu t\} = \nu(p^2 + \frac{3}{4})/R(p^2)$ and $\mathscr{L}\{\sinh\gamma t \cos \nu t\} = \gamma(p^2 - \frac{3}{4})/R(p^2)$.

Similarly, for Ψ, we have

$$\Psi_1 = \frac{p\mathscr{B}_1(p^2)}{(p^2 + \frac{9}{4})(p^2 + \frac{25}{4})R(p^2)} \quad \text{and} \quad \Psi_2 = \frac{p\mathscr{B}_2(p^2)}{(p^2 + \frac{9}{4})R(p^2)},$$

where

$$\mathscr{B}_1(p^2) = (5 + 4\nu)p^4 + \tfrac{7}{2}p^2(1 + 2\nu) - \tfrac{3}{16}(29 - 40\nu),$$
$$\mathscr{B}_2(p^2) = (1 + \nu)(4p^2 - 3)\delta_3 M_2 + \{p^4 + (1 - 2\delta_0)p^2 - \delta_1\}\psi_0$$
$$- \tfrac{3}{16}(1 + \nu)\{4(1 + 8\nu)p^2 - 3 + 40\nu\}M_1.$$

Splitting into partial fractions,

$$\Psi_1 = \frac{pB_3}{p^2+\frac{9}{4}} + \frac{pC_3}{p^2+\frac{25}{4}} + \frac{p(p^2-\gamma^2+v^2)D_3+2pE_3}{R(p^2)},$$

$$\Psi_2 = \frac{pB_4}{p^2+\frac{9}{4}} + \frac{p(p^2-\gamma^2+v^2)D_4+2pE_4}{R(p^2)},$$

where $B_3 = -\frac{16}{9}B_1$, $C_3 = -\frac{24}{25}C_1$, $B_4 = -\frac{16}{9}B_2$,

$$D_3 = \frac{11-13v}{3(1-v)(7-5v)}, \quad E_3 = \frac{v(1-2v)(15-17v)}{6(1-v)(7-5v)},$$

$$D_4 = \frac{1}{12(1-v)}\{16\delta_3 M_2 + 4(2-3v)\psi_0 - (3+8v)M_1\},$$

$$E_4 = \frac{-v}{24(1-v)}\{16v\delta_3 M_2 + 4(3+2v-6v^2)\psi_0 + (48-3v-56v^2)M_1\}.$$

Inverting the Laplace transforms, we obtain

$$\psi(t) = (\eta B_3 + B_4)\cos(\tfrac{3}{2}t) + \eta C_3\cos(\tfrac{5}{2}t) + (\eta D_3 + D_4)\cosh\gamma t\cos vt + (\gamma v)^{-1}(\eta E_3 + E_4)\sinh\gamma t\sin vt, \tag{6.9}$$

using $\mathscr{L}\{\cosh\gamma t\cos vt\} = p(p^2-\gamma^2+v^2)/R(p^2)$ and $\mathscr{L}\{\sinh\gamma t\sin vt\} = 2\gamma vp/R(p^2)$. Note that, setting $t=0$ in Eq. (6.9), we obtain $\psi(0)=\psi_0$, as expected, since $B_3+C_3+D_3=0$ and $B_4+D_4=\psi_0$.

6.1. Determination of M_1, M_2 and ψ_0

At this stage, we have expressions for $\varphi(t)$ and $\psi(t)$ involving the three constants ψ_0, M_1 and M_2. To determine these, we use the definitions of M_1 and M_2, Eq. (6.3), and constraint (5.4). We start by rewriting Eqs. (6.8) and (6.9) so as to display the three constants. Thus, we have

$$\varphi(t) = \eta\varphi_1(t) + (1-v)^{-1}\{\psi_0 h_0(t) + M_1 h_1(t) + M_2 h_2(t)\}, \tag{6.10}$$

$$\psi(t) = \eta\psi_1(t) + (1-v)^{-1}\{\psi_0 \ell_0(t) + M_1 \ell_1(t) + M_2 \ell_2(t)\}, \tag{6.11}$$

where

$$\varphi_1(t) = 2A_1\sin(\tfrac{1}{2}t) + \tfrac{2}{3}B_1\sin(\tfrac{3}{2}t) + \tfrac{2}{5}C_1\sin(\tfrac{5}{2}t) + v^{-1}D_1\cosh\gamma t\sin vt + \gamma^{-1}E_1\sinh\gamma t\cos vt,$$

$$\psi_1(t) = B_3\cos(\tfrac{3}{2}t) + C_3\cos(\tfrac{5}{2}t) + D_3\cosh\gamma t\cos vt + (\gamma v)^{-1}E_3\sinh\gamma t\sin vt,$$

$$h_0(t) = -\tfrac{1}{8}\{\sin(\tfrac{3}{2}t) + (3-4v)\cosh\gamma t\sin vt + \tfrac{1}{2}\gamma^{-1}(1+6v-8v^2)\sinh\gamma t\cos vt\},$$

$$h_1(t) = -\tfrac{1}{32}\{(3+8v)\sin(\tfrac{3}{2}t) + (13-24v)\cosh\gamma t\sin vt + \tfrac{1}{2}\gamma^{-1}(51-14v-48v^2)\sinh\gamma t\cos vt\},$$

$$h_2(t) = \tfrac{1}{2}\delta_2\{\sin(\tfrac{3}{2}t) - \cosh\gamma t\sin vt + \tfrac{1}{2}\gamma^{-1}(1-2v)\sinh\gamma t\cos vt\},$$

$$\ell_0(t) = \tfrac{1}{3}\{\cos(\tfrac{3}{2}t) + (2-3v)\cosh\gamma t\cos vt - \tfrac{1}{2}\gamma^{-1}(3+2v-6v^2)\sinh\gamma t\sin vt\},$$

$$\ell_1(t) = \tfrac{1}{12}\{(3+8v)(\cos(\tfrac{3}{2}t) - \cosh\gamma t\cos vt) - \tfrac{1}{2}\gamma^{-1}(48-3v-56v^2)\sinh\gamma t\sin vt\},$$

$$\ell_2(t) = -\tfrac{4}{3}\delta_2\{\cos(\tfrac{3}{2}t) - \cosh\gamma t\cos vt + \tfrac{1}{2}(v/\gamma)\sinh\gamma t\sin vt\}.$$

Then, Eqs. (5.4) and (6.3) give

$$\sum_{j=1}^{3} A_{ij}x_j = c_i, \quad i=1,2,3, \tag{6.12}$$

where $x_1 = M_1$, $x_2 = M_2$, $x_3 = \psi_0$,

$$c_1 = (1-\nu)\eta \int \varphi_1 \, \mathrm{d}t, \quad c_2 = (1-\nu)\eta \int (\varphi_1 \cos t + \psi_1 \sin t) \, \mathrm{d}t, \quad c_3 = -(1-\nu)\eta \int \psi_1 \cos \tfrac{1}{2}t \, \mathrm{d}t,$$

$$A_{11} = 1 - \nu - \int h_1 \, \mathrm{d}t, \quad A_{12} = -\int h_2 \, \mathrm{d}t, \quad A_{13} = -\int h_0 \, \mathrm{d}t,$$

$$A_{21} = -\int (h_1 \cos t + \ell_1 \sin t) \, \mathrm{d}t, \quad A_{22} = 1 - \nu - \int (h_2 \cos t + \ell_2 \sin t) \, \mathrm{d}t,$$
$$A_{23} = -\int (h_0 \cos t + \ell_0 \sin t) \, \mathrm{d}t,$$

$$A_{31} = \int \ell_1 \cos\left(\frac{1}{2}t\right) \mathrm{d}t, \quad A_{32} = \int \ell_2 \cos\left(\frac{1}{2}t\right) \mathrm{d}t, \quad A_{33} = \int \ell_0 \cos\left(\frac{1}{2}t\right) \mathrm{d}t$$

and all the integrals are over the range $0 \leqslant t \leqslant \alpha$; they are all elementary. Eq. (6.12) is a system of three simultaneous algebraic equations for M_1, M_2 and ψ_0. Then, φ and ψ are given by Eqs. (6.10) and (6.11), respectively, the components of the crack-opening displacement $[\boldsymbol{u}]$ are given by Eqs. (5.2) and (5.5), and the stress-intensity factors are given by Eq. (5.6).

Rather than giving the explicit solution of system (6.12) (which is straightforward but tedious), we obtain an approximate asymptotic solution for a shallow spherical cap.

7. The shallow spherical-cap crack

Suppose that the cap is shallow, which means that $c \to \infty$ and $\alpha \to 0$ with $a = c \sin \alpha$ fixed. Then, the stress-intensity factors are given by Eq. (5.6) in which α is small. We can solve the 3×3 system (6.12) in this limit. We have

$$\varphi_1(t) = t + O(t^3) \quad \text{and} \quad \psi_1(t) = \tilde{\psi} t^2 + O(t^4) \quad \text{as } t \to 0,$$

making use of Eq. (6.7) and $B_3 + C_3 + D_3 = 0$; the constant $\tilde{\psi}$ is given by

$$\tilde{\psi} = -\tfrac{9}{8}B_3 - \tfrac{25}{8}C_3 + \tfrac{1}{2}(\gamma^2 - \nu^2)D_3 + E_3 = \tfrac{1}{2}(5 + 4\nu).$$

There are similar approximations for $h_i(t)$ and $\ell_i(t)$, and these lead to small-α approximations for A_{ij} and c_i. Hence, we obtain

$$M_1 = M_2 = \tfrac{1}{2}\eta\alpha^2 + O(\alpha^4) \quad \text{and} \quad \psi_0 = -\tfrac{1}{3}\eta\tilde{\psi}\alpha^2 + O(\alpha^4)$$

as $\alpha \to 0$. It follows that

$$\varphi(\alpha) = \eta\alpha + O(\alpha^3) \quad \text{and} \quad \psi(\alpha) = \tfrac{2}{3}\eta\tilde{\psi}\alpha^2 + O(\alpha^4)$$

as $\alpha \to 0$, whence

$$K_n = \eta + O(\alpha^2) \quad \text{and} \quad K_s = -\tfrac{1}{6}\eta(5 + 4\nu)\alpha + O(\alpha^3) \quad \text{as } \alpha \to 0 \tag{7.1}$$

with $\eta = -4(1-\nu)p_0/(\pi\mu)$. The leading-order term corresponds to the well-known stress-intensity factor for a penny-shaped crack opened by a constant pressure; (Eq. (4.9)). The first-order correction is seen to occur in the tangential component. We have derived this first-order correction by an independent method, giving some credence to both approaches. In that method, we combined a perturbation expansion with an exact hypersingular boundary integral equation for $[\boldsymbol{u}]$, the method being designed for cracks that are perturbations of flat circular cracks, namely, wrinkled penny-shaped cracks (Martin, 2000).

Appendix A. The Mehler–Dirichlet integral, sums and variants

The standard Mehler–Dirichlet integral is (Whittaker and Watson, 1927, Section 15.231)

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})t}{\sqrt{2\cos t - 2\cos\theta}}\,\mathrm{d}t, \tag{A.1}$$

valid for $n = 0, 1, 2, \ldots$, and $0 \leqslant \theta \leqslant \pi$. Define $\lambda_n = n + \frac{1}{2}$ and

$$f_0(t;\theta) = \begin{cases} (2\cos t - 2\cos\theta)^{-1/2}, & 0 < t < \theta, \\ 0, & \theta < t \leqslant \pi, \end{cases} \tag{A.2}$$

so that $P_n(\cos\theta) = (2/\pi)\int_0^\pi f_0(t;\theta)\cos\lambda_n t\,\mathrm{d}t$. Define $f_0(t;\theta)$ for $\pi \leqslant t < 2\pi$ by $f_0(t;\theta) = -f_0(2\pi - t;\theta)$ and then expand the resulting extended function as a half-range Fourier cosine series. The result is

$$f_0(t;\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)\cos\lambda_n t, \quad 0 < t, \quad \theta \leqslant \pi, \quad t \neq \theta. \tag{A.3}$$

Such discontinuous sums are useful when solving dual series equations.

Replacing t and θ in Eq. (A.1) by $(\pi - t)$ and $(\pi - \theta)$, respectively, we obtain $P_n(\cos\theta) = (2/\pi)\int_0^\pi f_1(t;\theta)\sin\lambda_n t\,\mathrm{d}t$, where

$$f_1(t;\theta) = \begin{cases} 0, & 0 \leqslant t < \theta, \\ (2\cos\theta - 2\cos t)^{-1/2}, & \theta < t \leqslant \pi. \end{cases}$$

Extending f_1 using $f_1(t;\theta) = f_1(2\pi - t;\theta)$ for $\pi \leqslant t \leqslant 2\pi$ and then expanding as a half-range sine series gives

$$f_1(t;\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)\sin\lambda_n t, \quad 0 \leqslant t, \quad \theta < \pi, \quad t \neq \theta. \tag{A.4}$$

From Eqs. (A.3) and (A.4), we have

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\frac{\sin\lambda_n t}{\lambda_n} = \int_0^t f_0(\tau;\theta)\,\mathrm{d}\tau, \tag{A.5}$$

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\frac{\cos\lambda_n t}{\lambda_n} = \int_t^\pi f_1(\tau;\theta)\,\mathrm{d}\tau.$$

We also have

$$f_1(t;\theta)\cos t \pm f_0(t;\theta)\sin t = \sum_{n=0}^{\infty} P_n(\cos\theta)\sin(\lambda_n \pm 1)t,$$

$$f_0(t;\theta)\cos t \mp f_1(t;\theta)\sin t = \sum_{n=0}^{\infty} P_n(\cos\theta)\cos(\lambda_n \pm 1)t,$$

whence

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\frac{\sin(\lambda_n \pm 1)t}{\lambda_n \pm 1} = \int_0^t \{f_0(\tau;\theta)\cos\tau \mp f_1(\tau;\theta)\sin\tau\}\,\mathrm{d}\tau = \mathscr{S}_\pm,$$

and

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\,\frac{\cos(\lambda_n \pm 1)t}{\lambda_n \pm 1} = \int_t^{\pi} \{f_1(\tau;\theta)\,\cos\tau \pm f_0(\tau;\theta)\,\sin\tau\}\,\mathrm{d}\tau = \mathscr{C}_{\pm}.$$

Thus,

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\,\frac{\sin\lambda_n t}{\lambda_n \pm 1} = \mathscr{S}_{\pm}\cos t \mp \mathscr{C}_{\pm}\sin t, \sum_{n=0}^{\infty} P_n(\cos\theta)\,\frac{\cos\lambda_n t}{\lambda_n \pm 1} = \mathscr{C}_{\pm}\cos t \pm \mathscr{S}_{\pm}\sin t.$$

In particular,

$$\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n - 1} + \frac{1}{\lambda_n + 1}\right) P_n(\cos\theta)\,\sin\lambda_n t = (\mathscr{S}_+ + \mathscr{S}_-)\,\cos t - (\mathscr{C}_+ - \mathscr{C}_-)\,\sin t$$
$$= 2\int_0^t f_0(\tau;\theta)\,\cos(t-\tau)\,\mathrm{d}\tau - 4\sin t\,\sin\tfrac{1}{2}\theta,$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n - 1} - \frac{1}{\lambda_n + 1}\right) P_n(\cos\theta)\,\cos\lambda_n t = (\mathscr{C}_- - \mathscr{C}_+)\,\cos t - (\mathscr{S}_+ + \mathscr{S}_-)\,\sin t$$
$$= -2\int_0^t f_0(\tau;\theta)\,\sin(t-\tau)\,\mathrm{d}\tau - 4\cos t\,\sin\tfrac{1}{2}\theta.$$

We need similar formulas involving P_n^1. Use of the recurrence relation

$$(2n+1)P_n^1(\cos\theta)\,\sin\theta = n(n+1)\{P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)\}$$

gives

$$\frac{2n+1}{n(n+1)}P_n^1(\cos\theta)\,\sin\theta = \frac{4}{\pi}\int_0^{\pi} f_0(t;\theta)\,\sin t\,\sin\lambda_n t\,\mathrm{d}t$$
$$= \frac{-4}{\pi}\int_0^{\pi} f_1(t;\theta)\,\sin t\,\cos\lambda_n t\,\mathrm{d}t$$

for $n = 1, 2, \ldots$. It follows that

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{w_n}P_n^1(\cos\theta)\,\sin\lambda_n t = -\frac{1}{2}\tan\left(\frac{1}{2}\theta\right)\,\sin\left(\frac{1}{2}t\right) + \frac{\sin t}{\sin\theta}f_0(t;\theta) \tag{A.6}$$

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{w_n}P_n^1(\cos\theta)\,\cos\lambda_n t = \frac{1}{2}\cot\left(\frac{1}{2}\theta\right)\,\cos\left(\frac{1}{2}t\right) - \frac{\sin t}{\sin\theta}f_1(t;\theta) \tag{A.7}$$

for $0 < t, \theta < \pi$, $t \neq \theta$, where $w_n = n(n+1)$. Integrating Eq. (A.6) once gives

$$\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\cos\lambda_n t}{w_n} = -\tan\left(\frac{1}{2}\theta\right)\,\cos\left(\frac{1}{2}t\right) + \frac{f_0^1(t;\theta)}{\sin\theta},$$

where

$$f_0^1(t;\theta) = \begin{cases} (2\cos t - 2\cos\theta)^{1/2}, & 0 \leqslant t < \theta, \\ 0, & \theta \leqslant t \leqslant \pi. \end{cases} \tag{A.8}$$

Integrating again gives

$$\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\sin\lambda_n t}{\lambda_n w_n} = -2\tan\left(\frac{1}{2}\theta\right)\,\sin\left(\frac{1}{2}t\right) + \frac{1}{\sin\theta}\int_0^t f_0^1(\tau;\theta)\,\mathrm{d}\tau.$$

Similarly, Eq. (A.7) gives

$$\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\sin\lambda_n t}{w_n} = \cot\left(\frac{1}{2}\theta\right)\,\sin\left(\frac{1}{2}t\right) - \frac{f_1^1(t;\theta)}{\sin\theta},$$

where

$$f_1^1(t;\theta) = \begin{cases} 0, & 0 \leqslant t < \theta, \\ (2\cos\theta - 2\cos t)^{1/2}, & \theta \leqslant t \leqslant \pi. \end{cases}$$

We also have

$$\sin\theta\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\sin(\lambda_n \pm 1)t}{w_n} = (1+\cos\theta)\sin\left(\frac{1}{2}t\right)\,\cos t \mp (1-\cos\theta)\cos\left(\frac{1}{2}t\right)\,\sin t - f_1^1(t;\theta)\,\cos t \pm f_0^1(t;\theta)\,\sin t,$$

$$\sin\theta\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\cos(\lambda_n \pm 1)t}{w_n} = -(1-\cos\theta)\cos\left(\frac{1}{2}t\right)\,\cos t \mp (1+\cos\theta)\sin\left(\frac{1}{2}t\right)\,\sin t + f_0^1(t;\theta)\,\cos t \pm f_1^1(t;\theta)\,\sin t,$$

whence

$$\begin{aligned}\sin\theta\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\sin(\lambda_n \pm 1)t}{w_n(\lambda_n \pm 1)} &= \int_0^t \left\{f_0^1(\tau;\theta)\,\cos\tau \pm f_1^1(\tau;\theta)\,\sin\tau\right\}\mathrm{d}\tau - (1-\cos\theta)\biggl(\sin\left(\frac{1}{2}t\right) \\ &\quad + \frac{1}{3}\sin\left(\frac{1}{2}t\right)\biggr) \mp (1+\cos\theta)\biggl(\sin\left(\frac{1}{2}t\right) - \frac{1}{3}\sin\left(\frac{1}{2}t\right)\biggr) \\ &= \mathscr{S}_{\pm}^1,\end{aligned}$$

and

$$\begin{aligned}\sin\theta\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\cos(\lambda_n \pm 1)t}{w_n(\lambda_n \pm 1)} &= -\int_t^{\pi} \left\{f_1^1(\tau;\theta)\,\cos\tau \mp f_0^1(\tau;\theta)\,\sin\tau\right\}\mathrm{d}\tau + (1+\cos\theta)\biggl(-\cos\left(\frac{1}{2}t\right) \\ &\quad + \frac{1}{3}\cos\left(\frac{1}{2}t\right)\biggr) \mp (1-\cos\theta)\biggl(\cos\left(\frac{1}{2}t\right) + \frac{1}{3}\cos\left(\frac{1}{2}t\right)\biggr) \\ &= \mathscr{C}_{\pm}^1.\end{aligned}$$

Thus,

$$\sin\theta\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\sin\lambda_n t}{w_n(\lambda_n \pm 1)} = \mathscr{S}_{\pm}^1\cos t \mp \mathscr{C}_{\pm}^1\sin t,$$

$$\sin\theta\sum_{n=1}^{\infty} P_n^1(\cos\theta)\,\frac{\cos\lambda_n t}{w_n(\lambda_n \pm 1)} = \mathscr{C}_{\pm}^1\cos t \pm \mathscr{S}_{\pm}^1\sin t.$$

In particular,

$$\sin\theta\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n-1}+\frac{1}{\lambda_n+1}\right)P_n^1(\cos\theta)\,\frac{\sin\lambda_n t}{w_n}=\left(\mathscr{S}_+^1+\mathscr{S}_-^1\right)\cos t+\left(\mathscr{C}_-^1-\mathscr{C}_+^1\right)\sin t$$
$$=2\int_0^t f_0^1(\tau;\theta)\cos(t-\tau)\,\mathrm{d}\tau+\frac{16}{3}\sin^2\left(\frac{1}{2}\theta\right)\left(\frac{1}{2}\sin\left(\frac{1}{2}t\right)-\sin\left(\frac{1}{2}\theta\right)\sin t\right),$$

$$\sin\theta\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n-1}-\frac{1}{\lambda_n+1}\right)P_n^1(\cos\theta)\,\frac{\cos\lambda_n t}{w_n}=\left(\mathscr{C}_-^1-\mathscr{C}_+^1\right)\cos t-\left(\mathscr{S}_+^1+\mathscr{S}_-^1\right)\sin t$$
$$=-2\int_0^t f_0^1(\tau;\theta)\sin(t-\tau)\,\mathrm{d}\tau+\frac{16}{3}\sin^2\left(\frac{1}{2}\theta\right)\left(\cos\left(\frac{1}{2}t\right)-\sin\left(\frac{1}{2}\theta\right)\cos t\right).$$

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